Study of the Deterministic Component of the Solution Error of Discrete Ill-Posed Problems for Truncated Singular Value Decomposition and Random Projection Methods

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Abstract. *We have shown that the deterministic component of error obtained by the random projection method can be represented as the sum of the deterministic component of error obtained by the truncated singular value decomposition method and another term.*

Keywords

ill-posed inverse problems, truncated singular value decomposition, random projection.

1 Introduction

Discrete ill-posed problems (DIP) often occur when signals are reconstructed based on the results of indirect measurements [1, 2, 3]. The stable solution of DIP can be obtained, for example, by truncated singular value decomposition (SVD) [4, 5, 6, 7] or random projection [8, 9, 10, 11] methods.

The studying of the solution error for these methods is actual. In this paper we have taken steps to study the connection of the deterministic component of the solution error obtained by the truncated singular value decomposition and the random projection methods.

2 Solving of DIP using the methods of truncated singular value decomposition and random projection

The problem of estimating a signal vector $\mathbf{x} \in \Re^n$ for the model $\mathbf{y} = \mathbf{Ax} + \boldsymbol{\varepsilon}$, where we know the matrix $\mathbf{A} \in \Re^{n\times n}$ and the vector of measurements $\mathbf{y} \in \Re^n$ ($\mathbf{y} = \mathbf{y}_0 + \boldsymbol{\varepsilon}$), if the singular values of the matrix $\mathbf{A}$ gradually decrease to zero, belongs to the class of discrete ill-posed problems [7].

Stable solution of DIP can be obtained by using the truncated singular value decomposition [4, 5, 6, 7] or the random projection method [8, 9, 10, 11]. The vector of estimation $\mathbf{x}^*$ based on truncated singular value decomposition is obtained by the following linear model [6]:

$$\mathbf{x}^*_{k\ SVD} = \sum_{i=1}^{k} \mathbf{v}_i s_i^{-1} \mathbf{u}_i^{\mathrm{T}} \mathbf{y}\,, \tag{1}$$

where $\mathbf{u}_i \in \Re^n$ are left singular vectors, $\mathbf{v}_i \in \Re^n$ are right singular vectors and s_i are singular values of the matrix $\mathbf{A}$. Note that $\mathbf{A} = \mathbf{A}_k + \mathbf{A}_n$, where $\mathbf{A}_k = \sum_{i=1}^{k} \mathbf{u}_i s_i \mathbf{v}_i^{\mathrm{T}}$, $\mathbf{A}_n = \sum_{i=k+1}^{n} \mathbf{u}_i s_i \mathbf{v}_i^{\mathrm{T}}$.

For the solution based on random projection both sides of the given equation are multiplied by a matrix $\mathbf{G}_k \in \Re^{k\times n}$: $\mathbf{G}_k\mathbf{Ax} = \mathbf{G}_k\mathbf{y}$, and then the estimation vector $\mathbf{x}^*$ is obtained by the following linear model [8]:

$$\mathbf{x}^*_{k\ G} = \sum_{i=1}^{k}\mathbf{h}_i\mathbf{g}_i^{\mathrm{T}}\mathbf{y}\,, \tag{2}$$

where $\mathbf{g}_i \in \Re^n$ is a column of the matrix $\mathbf{G}_k = [\mathbf{g}_1,...,\mathbf{g}_k]$, $\mathbf{G}_k \in \Re^{k\times n}$, elements of this matrix are realizations of a standard Gaussian random variable; $\mathbf{h}_i \in \Re^n$ is a column of the matrix $(\mathbf{G}_k\mathbf{A})^+ = [\mathbf{h}_1,...,\mathbf{h}_k]$.

The expected squared error of the $\mathbf{x}$ recovery is calculated as $e(k) = \mathrm{E}\{\left\|\mathbf{x}-\mathbf{x}^*_k\right\|^2\}$, where $\mathrm{E}\{\cdot\}$ is expectation with respect to noise in the vector of measurements y and $\mathbf{x}^*_k$ is the recovered signal vector obtained by (1) or (2) with $\mathbf{y}$ obtained by some noise realization.

The expected squared error for the truncated SVD obtained in [4, 5] is

$$e_{SVD}(k) = \left\|(\mathbf{A}^+_k\mathbf{A}_k - \mathbf{I})\mathbf{x}\right\|^2 + \sigma^2 trace\,(\mathbf{A}^{+\mathrm{T}}_k\mathbf{A}^+_k) \tag{3}$$

and its components are

$$e_{SVD\ d}(k) = \left\|(\mathbf{A}^+_k\mathbf{A}_k - \mathbf{I})\mathbf{x}\right\|^2 \text{ and } e_{SVD\ s}(k) = \sigma^2 trace\,(\mathbf{A}^{+\mathrm{T}}_k\mathbf{A}^+_k)\,, \tag{4}$$

where $e_{x\ d}$ is the deterministic component of the error and $e_{x\ s}$ is the stochastic component.

The expected squared error for the random projection method obtained in [8] is

$$e_G(k) = \left\|(\mathbf{B}^+_k\mathbf{B}_k - \mathbf{I})\mathbf{x}\right\|^2 + \sigma^2 trace\,(\mathbf{B}^{+\mathrm{T}}_k\mathbf{B}^+_k),\ \mathbf{B}_k = (\mathbf{G}_k\mathbf{A}) \in \Re^{k\times n}\,, \tag{5}$$

and its components are

$$e_{G\ d}(k) = \left\|(\mathbf{B}^+_k\mathbf{B}_k - \mathbf{I})\mathbf{x}\right\|^2,\ e_{G\ s}(k) = \sigma^2 trace\,(\mathbf{B}^{+\mathrm{T}}_k\mathbf{B}^+_k)\,. \tag{6}$$

Numerous experimental studies [8, 9] of the dependence of expected squared error estimation on the number k of model components for the truncated SVD and for the random projection methods showed closeness of these dependencies.

In this paper we study closeness of $e_{SVD\ d}(k)$ and $e_{G\ d}(k)$ analytically.

3 The study of deterministic component of error for the truncated singular value decomposition and for the random projection methods

Let us study the connection between the deterministic component of error for the truncated SVD and for the random projection methods. $\mathbf{B}_k$ can be represented as the sum: $\mathbf{B}_k = \mathbf{G}_k\mathbf{A} = \mathbf{G}_k\mathbf{A}_k + \mathbf{G}_k\mathbf{A}_n$. We consider $\mathbf{G}_k\mathbf{A}_n$ as the disturbance of the matrix $\mathbf{G}_k\mathbf{A}_k$ and represent $\mathbf{B}^+_k$ by the Stewart formula [12]:

$$\mathbf{B}^+_k = \mathbf{C}^+_k + (\mathbf{G}_k\mathbf{A})^+\mathbf{P}_k(\mathbf{I}-\mathbf{C}_k\mathbf{C}^+_k) + (\mathbf{G}_k\mathbf{A})^+\mathbf{P}_k\mathbf{E}_k\mathbf{D}_k\mathbf{C}^+_k - \mathbf{R}^\perp_k\mathbf{D}_k\mathbf{C}^+_k\,, \tag{7}$$

where $\mathbf{C}_k = \mathbf{G}_k\mathbf{A}_k$, $\mathbf{E}_k = \mathbf{G}_k\mathbf{A}_n$, $\mathbf{D}_k = \mathbf{C}^+_k\mathbf{C}_k$, $\mathbf{R}^\perp_k = \mathbf{I} - (\mathbf{G}_k\mathbf{A})^+\mathbf{G}_k\mathbf{A}$.

Taking into account that $\mathbf{C}_k\mathbf{C}^+_k = \mathbf{I}_k$, $\mathbf{P}_k = \mathbf{G}_k\mathbf{A}(\mathbf{G}_k\mathbf{A})^+ = \mathbf{I}_k$, $\mathbf{E}_k\mathbf{C}^+_k = \mathbf{O}$, $(\mathbf{I}-\mathbf{C}_k\mathbf{C}^+_k) = \mathbf{O}$, where $\mathbf{O}$ is a zero matrix, we obtain the following equation

$$\mathbf{B}_k^+ = \mathbf{C}_k^+ - \mathbf{R}_k^{\perp}\mathbf{C}_k^+ . \tag{8}$$

Using (8) and taking into account that $\mathbf{B}_k = \mathbf{C}_k + \mathbf{E}_k$, the expression for the deterministic error component can be written as:

$$e_{G\ d}(k) = \left\|(\mathbf{B}_k^+\mathbf{B}_k - \mathbf{I})\mathbf{x}\right\|^2 = \left\|((\mathbf{C}_k^+ - \mathbf{R}_k^{\perp}\mathbf{C}_k^+)\mathbf{B}_k - \mathbf{I})\mathbf{x}\right\|^2 = \left\|(\mathbf{C}_k^+\mathbf{B}_k - \mathbf{I})\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\|^2,$$

$$e_{G\ d}(k) = \left\|(\mathbf{C}_k^+(\mathbf{C}_k + \mathbf{E}_k) - \mathbf{I})\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\|^2 = \left\|(\mathbf{D}_k + \mathbf{C}_k^+\mathbf{E}_k - \mathbf{I})\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\|^2 . \tag{9}$$

Using the SVD of $\mathbf{A}_k$, the $\mathbf{D}_k$ matrix can be represented as

$$\mathbf{C}_k^+\mathbf{C}_k = (\mathbf{G}_k\mathbf{A}_k)^+\mathbf{G}_k\mathbf{A}_k = (\mathbf{G}_k\mathbf{U}_k\mathbf{S}_k\mathbf{V}_k^{\mathrm{T}})^+\mathbf{G}_k\mathbf{A}_k = (\mathbf{S}_k\mathbf{V}_k^{\mathrm{T}})^+(\mathbf{G}_k\mathbf{U}_k)^+\mathbf{G}_k\mathbf{U}_k\mathbf{S}_k\mathbf{V}_k^{\mathrm{T}} . \tag{10}$$

Since $(\mathbf{G}_k\mathbf{U}_k)^+\mathbf{G}_k\mathbf{U}_k = \mathbf{I}_k$

$$\mathbf{D}_k = (\mathbf{S}_k\mathbf{V}_k^{\mathrm{T}})^+\mathbf{S}_k\mathbf{V}_k^{\mathrm{T}} = \mathbf{A}_k^+\mathbf{A}_k . \tag{11}$$

Note that after (10, 11), the matrix $\mathbf{G}_k$ (with the elements that are realizations of a standard Gaussian random variable) is not used in the representation of $\mathbf{D}_k$. This allows us to represent the deterministic error component of the random projection method as a sum including the deterministic error component of the truncated SVD method:

$$e_{G\ d}(k) = \left\|(\mathbf{A}_k^+\mathbf{A}_k - \mathbf{I})\mathbf{x} + \mathbf{C}_k^+\mathbf{E}_k\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\|^2 , \tag{12}$$

$$e_{G\ d}(k) = \left\|(\mathbf{A}_k^+\mathbf{A}_k - \mathbf{I})\mathbf{x}\right\|^2 + \left\|\mathbf{C}_k^+\mathbf{E}_k\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\|^2 + 2\left\langle(\mathbf{A}_k^+\mathbf{A}_k - \mathbf{I})\mathbf{x}, \mathbf{C}_k^+\mathbf{E}_k\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\rangle . \tag{13}$$

$$e_{G\ d}(k) = e_{SVD\ d}(k) + z(k); \tag{14}$$

where $z(k) = \left\|\mathbf{C}_k^+\mathbf{E}_k\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\|^2 + 2\left\langle(\mathbf{A}_k^+\mathbf{A}_k - \mathbf{I})\mathbf{x}, \mathbf{C}_k^+\mathbf{E}_k\mathbf{x} - \mathbf{R}_k^{\perp}\mathbf{C}_k^+\mathbf{B}_k\mathbf{x}\right\rangle$. The expression (14) provides a background for studying the value of the difference $|e_{G\ d}(k) - e_{SVD\ d}(k)|$, for example, by estimating the upper bound of $z(k)$. Figures 1-4 show the examples of dependencies of $e_{SVD\ d}(k)$ and $e_{G\ d}(k)$ for the Phillips, Delves, Carasso and Baart problems [13].

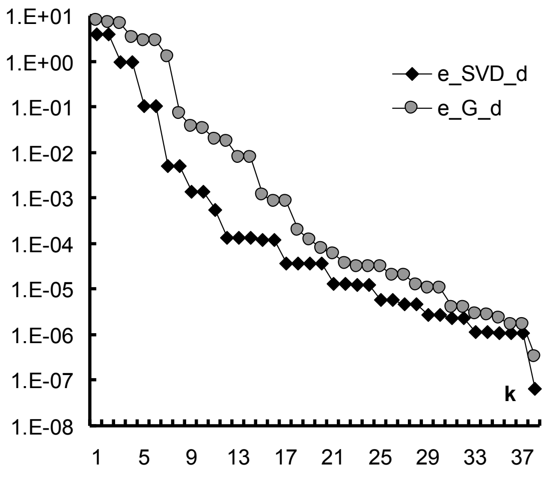

Fig.1. Dependencies of $e_{SVD\ d}(k)$ and $e_{G\ d}(k)$ for the Phillips problem

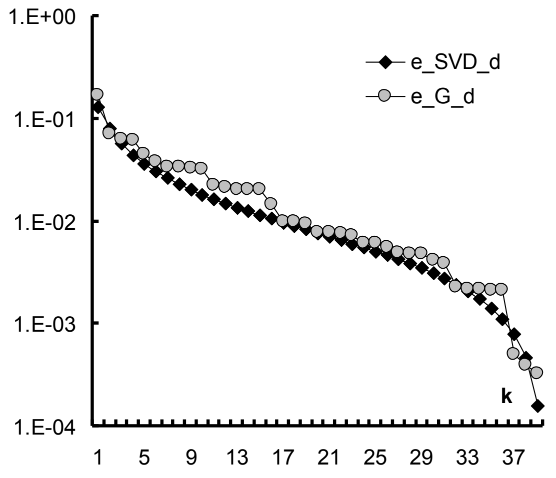

Fig.2. Dependencies of $e_{SVD\ d}(k)$ and $e_{G\ d}(k)$ for the Delves problem

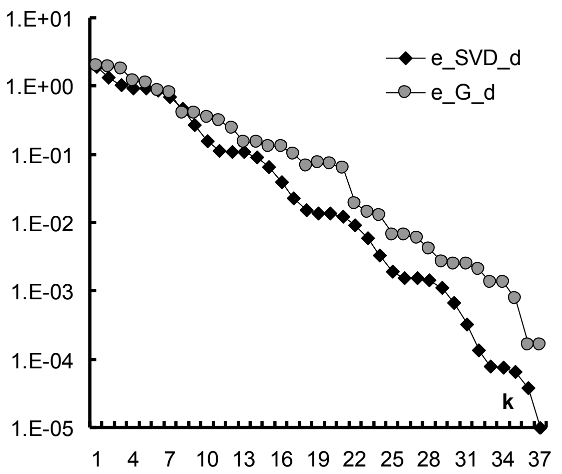

Fig.3. Dependencies of $e_{SVD\ d}(k)$ and $e_{G\ d}(k)$ for the Carasso problem

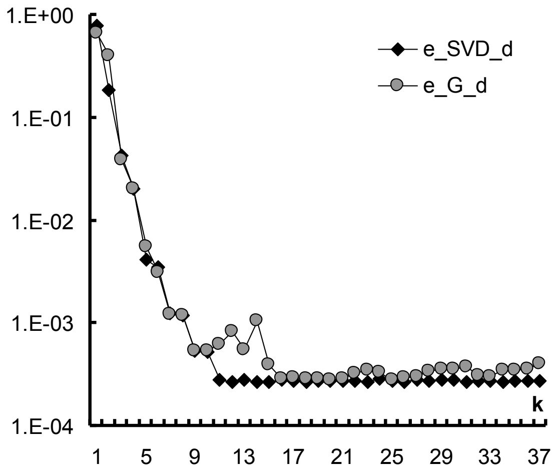

Fig.4. Dependencies of $e_{SVD\ d}(k)$ and $e_{G\ d}(k)$ for the Baart problem

4 Conclusion

In the previous papers we experimentally investigated the dependence of the mean squared error of signal recovery on the number k of components in the linear model that represents the solution vector, for both the truncated singular value decomposition and the random projection methods. The experiments had shown that the dependences are close and motivated the analytical investigation of the reasons for such a closeness started in this paper.

We have shown that the deterministic component of error obtained by the random projection method can be represented as the sum of the deterministic component of error obtained by the truncated singular value decomposition method and another term. The direction of further analytical study is to get an upper bound for the additive term.

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